

# FAST RADIO BURSTS' RECIPES FOR THE DISTRIBUTIONS OF DISPERSION MEASURES, FLUX DENSITIES, AND FLUENCES

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## ABSTRACT

We investigate how the statistical properties of dispersion measure (DM) and apparent flux density/fluence of (non-repeating) fast radio bursts (FRBs) are determined by unknown cosmic rate density history  $[\rho_{\text{FRB}}(z)]$  and luminosity function (LF) of the transient events. We predict the distributions of DMs, flux densities, and fluences of FRBs taking account of the variation of the receiver efficiency within its beam, using analytical models of  $\rho_{\text{FRB}}(z)$  and LF. Comparing the predictions with the observations, we show that the cumulative distribution of apparent fluences suggests that FRBs originate at cosmological distances and  $\rho_{\text{FRB}}$  increases with redshift resembling cosmic star formation history (CSFH). We also show that a LF model with a bright-end cutoff at  $\log_{10} L_{\nu}$  [erg s<sup>-1</sup>Hz<sup>-1</sup>]  $\sim 34$  are favored to reproduce the observed DM distribution if  $\rho_{\text{FRB}}(z) \propto \text{CSFH}$ , although the statistical significance of the constraints obtained with the current size of the observed sample is not high. Finally, we find that the correlation between DM and flux density of FRBs is potentially a powerful tool to distinguish whether FRBs are at cosmological distances or in the local universe more robustly with future observations.

*Keywords:* radio continuum: general — intergalactic medium — ISM: general — methods: statistical

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## 1. INTRODUCTION

Fast Radio Bursts (FRBs) are transient events observed in  $\sim 1$  GHz radio bands with typical durations of several milliseconds (e.g., Lorimer et al. 2007; Keane et al. 2012; Thornton et al. 2013). Their large dispersion measures (DMs), which indicate the column density of free electrons along the sightlines, suggest that they are extragalactic objects. If FRB DMs arise from the intergalactic medium (IGM), FRBs may provide us with an unprecedented opportunity to detect the IGM directly.

However, the origin of FRBs is not known yet. Although various theoretical models have been proposed (e.g., Totani 2013; Kashiyama et al. 2013; Popov & Postnov 2013; Falcke & Rezzolla 2014; Cordes & Wasserman 2016; Zhang 2017), observational evidence that confirms or rejects those models is hardly obtained. The currently available localization precision of FRBs are typically several arcmin, which is too large to identify their host galaxies, and FRB distance measurements which are independent of DM are also missing.

The only exception is FRB 121102, the repeating FRB, for which the host galaxy is identified and its redshift is known thanks to its repeatability (Chatterjee et al. 2017; Tendulkar et al. 2017). However, the other FRBs do not show any repeatability, and hence FRB 121102 can be a different kind of phenomenon from the other FRBs (Palaniswamy & Zhang 2017), although it is also pointed out that FRB 110220 and FRB 140514 might be repetitions of a same source (Piro & Burke-Spolaor 2017). Hereafter, FRB means non-repeating FRB, unless stated otherwise.

Redshift distribution of a population of transient events is an important clue to understand the nature of the transients, which reflects their luminosity function and comoving rate density at each redshift. The cosmic FRB rate density [ $\rho_{\text{FRB}}(z)$ ] would be proportional to the cosmic star formation history (CSFH) if FRBs are related with young stellar population (e.g., core-collapse supernovae, young neutron stars), while it would follow the cosmic stellar mass density (CSMD) if FRBs arise from older stars.

Although we can not measure redshift of an FRB in most cases, distance to an FRB can be estimated via its DM. The excess of the DM over the Milky Way contribution in the direction ( $\text{DM}_{\text{EX}}$ ) can be interpreted as the distance to the source under the assumption that the major part of the observed  $\text{DM}_{\text{EX}}$  arise from the IGM (e.g., Ioka 2003; Inoue 2004). Previous studies have shown that the  $\text{DM}_{\text{EX}}$  distribution of the observed FRBs is consistent with the expectations if FRBs are distributed over cosmological distance (e.g., Dolag et al.

2015; Katz 2016; Caleb et al. 2016; Cao et al. 2017). However,  $\text{DM}_{\text{EX}}$  does not necessarily arise only from the IGM, because a part of  $\text{DM}_{\text{EX}}$  can be attributed to ionized gas in the vicinity of FRBs.

The cumulative distribution of FRB flux densities/fluences (so called  $\log N$ - $\log S$  distribution) also serves as a clue to understand the distance distribution of FRBs, because the distribution follows a power-law with the index of -1.5 when the sources are uniformly distributed in an Euclidian space while the distribution may vary when the sources are at cosmological distances due to the cosmic expansion and cosmological evolution of the source number density (Katz 2016; Vedantham et al. 2016; Caleb et al. 2016; Oppermann et al. 2016; Li et al. 2017; Macquart & Ekers 2018).

In this study, we investigate how the interplay between unknown cosmic rate density history and luminosity function of FRBs determines the statistical properties of the observable quantities, i.e.,  $\text{DM}_{\text{EX}}$  and apparent flux density/fluence, taking account of the variation of the receiver efficiency within its beam. We discuss what constraint the current observations put on the nature of FRBs, and how can we distinguish whether FRBs are at cosmological distances or in the local universe with future observations.

In §2 and §3, we describe our models of cosmic FRB rate history and FRB luminosity function (LF), respectively. We discuss constraints on the cosmic rate history and the LF of FRBs obtained from the observed  $\text{DM}_{\text{EX}}$  distribution under the assumption that FRBs originate at cosmological distances in §4. In §5, we discuss the  $\log N$ - $\log S$  distribution and the correlation between  $\text{DM}_{\text{EX}}$  and apparent flux density of FRBs, comparing the predictions of the cosmological and local FRB models. In §6, we discuss a couple of uncertainties that may potentially affect our results. Our conclusions are summarized in §7. Throughout this paper, we assume the fiducial cosmology with  $\Omega_{\Lambda} = 0.7$ ,  $\Omega_m = 0.3$ , and  $H_0 = 70 \text{ km s}^{-1} \text{ Mpc}^{-1}$ .

## 2. COSMIC FRB RATE HISTORY AND $\text{DM}_{\text{IGM}}$ DISTRIBUTION

We consider three models of  $\rho_{\text{FRB}}$  in this study (the top panel of figure 1). One is proportional to CSFH (SFR model), another is constant throughout cosmic time (constant model), and the other is proportional to CSMD ( $M_{\star}$  model). We use the formulations of CSFH and CSMD by Madau & Dickinson (2014).

The redshift distribution of FRBs that occur in a unit area on the sky within a certain time period in the ob-

server frame can be expressed as

$$\frac{dN(z)}{dzd\Omega} = \frac{\rho_{\text{FRB}}(z)}{1+z} \times \frac{dV}{dzd\Omega}, \quad (1)$$

where  $dV/dzd\Omega$  is comoving volume per redshift per observed area (the middle panel of figure 1).

DM that arise from the IGM can be expressed as:

$$\text{DM}_{\text{IGM}} = c \int_0^z \left| \frac{dt}{dz'} \right| \frac{n_{e,\text{IGM}}(z')}{1+z'} dz'. \quad (2)$$

where  $n_{e,\text{IGM}}$  is the electron density in the IGM. Here we assume that the IGM is uniform at each redshift with the comoving density  $\rho_{\text{crit}}\Omega_b$ , composed of 75% H and 25% He, and fully ionized throughout the redshift range we consider. Under these assumptions, the IGM electron density can be written as:

$$n_{e,\text{IGM}}(z) = \frac{7}{8} \frac{\rho_{\text{crit}}\Omega_b}{m_p} (1+z)^3. \quad (3)$$

The upper horizontal axis of figure 1 indicates  $\text{DM}_{\text{IGM}}$  that corresponds to  $z$  in the lower axis (naively  $\text{DM}_{\text{IGM}} \sim 1000z \text{ cm}^{-3}\text{pc}$  in this redshift range).

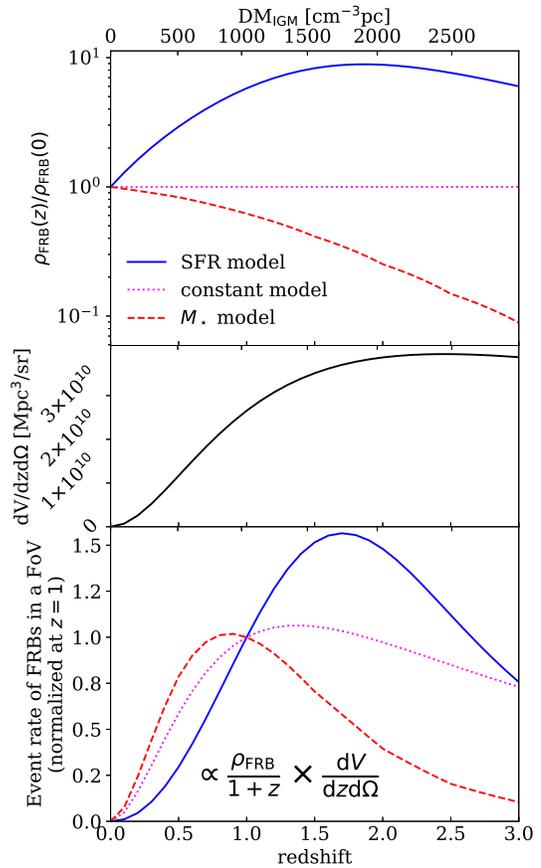
In the above expression, it is assumed that the dominant fraction of baryons in the universe is in the IGM, which is true when we consider diffuse ionized gas associated with dark matter halos as part of the IGM (e.g., Fukugita & Peebles 2004). If a significant part of the IGM is associated dark matter halos, the IGM might be inhomogeneous in reality, and the inhomogeneity might affect the  $\text{DM}_{\text{IGM}}$  distribution of FRBs. We discuss the effect of the IGM inhomogeneity on our results in §6.1

The predicted redshift distributions with the three  $\rho_{\text{FRB}}(z)$  models are shown in the bottom panel of figure 1. The redshift distributions with the different  $\rho_{\text{FRB}}(z)$  models are similar with each other at  $z \lesssim 1$  where majority of the currently known FRBs reside, while the redshift distributions are dramatically different at  $z > 1$ , as previously shown by Dolag et al. (2015) using cosmological simulations. We note that detectability of FRB events are not considered here and the redshift distributions may include FRBs that are too faint to be detected. We discuss fraction of detectable FRBs at each redshift in §3.

### 3. FRB LUMINOSITY AND RECEIVER EFFICIENCY

#### 3.1. Receiver efficiency variation within a beam

Observed radio flux density of an FRB at the peak of its light curve ( $S_{\nu,\text{app}}$ ) depends not solely on its luminosity and distance, but also on the unknown position of the FRB within the receiver beam, because efficiency

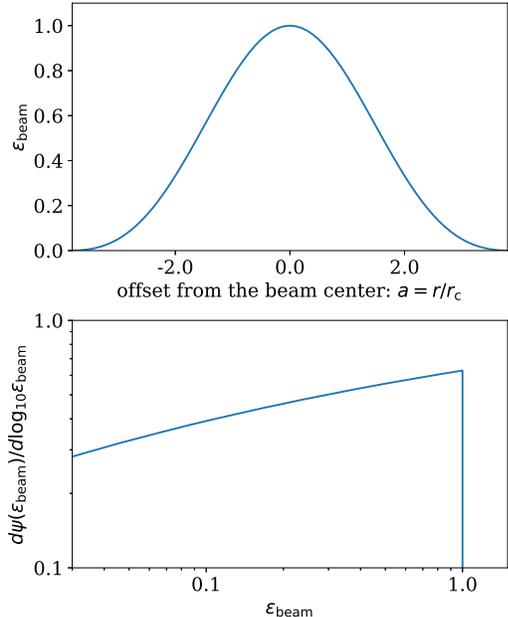


**Figure 1.** Top panel:  $\rho_{\text{FRB}}$  models (occurrence rate of FRBs per comoving volume) normalized at  $z = 0$ . Middle panel: observed comoving volume per redshift per steradian with the assumed cosmology ( $dV/dzd\Omega$ ). Bottom panel: occurrence rate of FRBs per redshift per steradian in the observer frame, which is proportional to  $\rho_{\text{FRB}}(z)/(1+z) \times dV/dzd\Omega$ . We note that the FRB rates shown in this figure represents all FRBs regardless of their detectability.  $\text{DM}_{\text{IGM}}$  that corresponds to each redshift is indicated in the upper horizontal axis (see equation 2).

of a radio receiver largely varies within its beam. We assume beam efficiency pattern of a radio receiver under consideration is represented by an Airy disc

$$\epsilon_{\text{beam}}(a) = \left[ \frac{2J_1(a)}{a} \right]^2, \quad (4)$$

where the efficiency at the beam center is unity,  $J_1$  is the first order Bessel function of the first kind, and  $a = r/r_c$  is the offset from the beam center normalized by the beam characteristic radius (the top panel of figure 2). The efficiency is 50% at  $a = 1.62$  and drops to zero at  $a = 3.83$  ( $\equiv a_{\text{out}}$ ). For the Parkes multi-beam receiver (Staveley-Smith et al. 1996) whose full width at half maximum (FWHM) is 14.4 arcmin,  $r_c$  is 4.4 arcmin. We do not consider sidelobe detections ( $|a| > a_{\text{out}}$ ).



**Figure 2.** Top panel: the Airy disk model of a radio receiver beam efficiency as a function of the offset from the beam center. Bottom panel: the efficiency PDF within an Airy disc beam.

The probability distribution function (PDF) of  $\epsilon_{\text{beam}}$  within a beam ( $|a| \leq a_{\text{out}}$ ) can be written as

$$\frac{d\psi(\epsilon_{\text{beam}})}{d\log_{10}\epsilon_{\text{beam}}} = \frac{2\ln 10}{a_{\text{out}}^2} \epsilon_{\text{beam}} a(\epsilon_{\text{beam}}) \left| \frac{da}{d\epsilon_{\text{beam}}} \right| \quad (5)$$

where  $a(\epsilon_{\text{beam}})$  is the inverse function of equation (4) in the range  $a > 0$ . In the bottom panel of figure 2, we show the PDF defined by equation (5). We note that the PDF is not dependent of the choice of  $r_c$ , and hence applicable to any radio telescope whose efficiency profile can be represented by an Airy disc.

### 3.2. Propagation effects and $K$ -correction

Flux density of a FRB is also affected by its propagation medium. While scattering may suppress FRB flux density by pulse broadening, scintillation and plasma lensing may also enhance FRB flux density (e.g., Hassall et al. 2013; Cordes et al. 2016, 2017). Currently it is difficult to separate the intrinsic LF of FRBs from the PDF of the propagation effects. In this study, we consider effective luminosity ( $L_{\nu, \text{eff}}$ ) which includes the propagation effects ( $\epsilon_{\text{prop}}$ ) rather than intrinsic luminosity ( $L_{\nu, \text{int}}$ ) of an FRB. We also consider apparent luminosity ( $L_{\nu, \text{app}}$ ) which includes  $\epsilon_{\text{beam}}$  in addition to  $\epsilon_{\text{prop}}$  and can be directly related to  $S_{\nu, \text{app}}$ .

$K$ -correction is also an important effect when we consider observed flux densities of objects at cosmological

distances. We express the  $K$ -correction factor as:

$$\kappa_{\nu}(z) = \frac{L_{\nu}(\nu_{\text{rest}})}{L_{\nu}(\nu_{\text{obs}})}, \quad (6)$$

where  $\nu_{\text{obs}}$  is the observing frequency and  $\nu_{\text{rest}} = (1 + z)\nu_{\text{obs}}$ . In the case of the Parkes multi-beam receiver,  $\nu_{\text{obs}} = 1.4$  GHz. The functional form of  $\kappa_{\nu}(z)$  is determined by spectra of FRBs which is not known yet. Here we assume  $\kappa_{\nu}(z) = 1$  (constant), and discuss how our results are affected by  $K$ -correction in §6.2.

In summary,

$$S_{\nu, \text{app}}(\nu_{\text{obs}}) = \frac{1 + z}{4\pi d_L(z)^2} \kappa_{\nu}(z) L_{\nu, \text{app}}(\nu_{\text{obs}}), \quad \text{and} \quad (7)$$

$$L_{\nu, \text{app}}(\nu_{\text{obs}}) = \epsilon_{\text{beam}} L_{\nu, \text{eff}}(\nu_{\text{obs}}) \quad (8)$$

$$= \epsilon_{\text{beam}} \epsilon_{\text{prop}} L_{\nu, \text{int}}(\nu_{\text{obs}}), \quad (9)$$

where  $d_L(z)$  is luminosity distance at redshift  $z$ .

### 3.3. FRB luminosity function

We examine the following three  $L_{\nu, \text{eff}}$  distribution function models to demonstrate how difference of FRB LF affects the observable properties of FRBs.

- LF1: FRBs are standard candles with  $L_{\nu, \text{eff}} = L_{\nu, 0}$ .
- LF2:  $L_{\nu, \text{eff}}$  follows a power-law distribution,  $d\phi(L_{\nu, \text{eff}})/dL_{\nu, \text{eff}} \propto L_{\nu, \text{eff}}^{-2}$ , with a faint-end cutoff at  $L_{\nu, 0}$ .
- LF3:  $L_{\nu, \text{eff}}$  follows a distribution function with the faint-end power-law index  $-1$  down to  $\log_{10} L_{\nu, \text{eff}} [\text{erg s}^{-1} \text{Hz}^{-1}] = 30.0$ , and exponential cutoff in the bright-end,  $L_{\nu, \text{eff}} \gtrsim L_{\nu, 0}$ , i.e.,  $d\phi(L_{\nu, \text{eff}})/dL_{\nu, \text{eff}} \propto x^{-1} \exp(-x)$ , where  $x = L_{\nu, \text{eff}}/L_{\nu, 0}$ .

The three  $L_{\nu, \text{eff}}$  PDFs and the corresponding  $L_{\nu, \text{app}}$  PDFs are shown in figure 3. The  $L_{\nu, \text{app}}$  PDFs are obtained by convoluting the  $L_{\nu, \text{eff}}$  PDFs with the  $\epsilon_{\text{beam}}$  PDF (equation 5). The faint-end cutoff of LF3 is adopted so that the integral of the LF is finite. The cutoff luminosity can be observed at redshifts only up to  $z \sim 0.01$  with the Parkes multi-beam receiver, and hence it is faint enough not to affect our result.

Although the shape of the bright-end of the  $L_{\nu, \text{app}}$  PDFs resembles that of the  $L_{\nu, \text{eff}}$  PDFs, the faint-end of the  $L_{\nu, \text{app}}$  PDFs is smeared out by the  $\epsilon_{\text{beam}}$  variation. Hence it will be difficult to constrain the faint-end of the  $L_{\nu, \text{eff}}$  PDFs from the currently observable properties of FRBs. Although the actual shape of the FRB LF is hardly known, we consider the three LF models described above can represent a wide variety of LFs due to this smearing. We note that the PDF of  $\epsilon_{\text{prop}}$ , and

hence the  $L_{\nu,\text{eff}}$  PDF, may depend on galactic latitude and longitude of observation fields, if the propagation effects in the Milky Way significantly affect observed flux densities. However, we assume that all FRBs under consideration arise from a single  $L_{\nu,\text{eff}}$  PDF and consider the PDF as the average of those in all observation fields.

### 3.4. Detection of an FRB

To constrain the FRB models, we use the sample of FRBs detected by the Parkes multi-beam receiver before the end of 2017 November (21 FRBs between 010125 and 160102). The properties of the observed FRBs are taken from the FRBCAT<sup>1</sup> (Petroff et al. 2016). Although the values in the FRBCAT are derived separately by individual authors, Petroff et al. (2016) have reanalyzed some of the data in a uniform manner, and we use the values derived by the reanalysis when available. We note that FRBs discovered by different telescopes should not be treated together in a statistical study of the DM distribution because the  $\text{DM}_{\text{IGM}}$  distribution of a sample of FRBs would depend on the detection limit of the observations.

We compute the fraction of detectable FRBs at each redshift using the  $L_{\nu,\text{app}}$  distribution functions. For simplicity, we consider an FRB is detected when the apparent flux density exceeds a threshold,  $S_{\nu,\text{app}} \geq S_{\nu,\text{th}}$ . To compare our model predictions with the Parkes detected FRB sample, we assume the threshold flux density  $S_{\nu,\text{th}} = 0.4$  Jy which is comparable to the faintest FRBs in the Parkes sample.

Although it is pointed out that detectability of an FRB depends not only on its flux but also on the pulse width (and hence the fluence, Keane & Petroff 2015), the Parkes sample shows that  $S_{\nu,\text{app}}$  is a better proxy for signal-to-noise ratio (S/N) than apparent fluence [observed fluence including  $\epsilon_{\text{beam}}(F_{\nu,\text{app}})$ , see figure 4]. When the saturated event FRB 010724 (Lorimer et al. 2007) and the extremely bright outlier event FRB 150807 (Ravi et al. 2016) are excluded from the sample, the correlation coefficient between  $\log_{10}S_{\nu,\text{app}}$  and  $\log_{10}\text{S/N}$  is 0.79 (0.58 between  $\log_{10}F_{\nu,\text{app}}$  and  $\log_{10}\text{S/N}$ ). In figure 5, we show how the predicted  $\text{DM}_{\text{IGM}}$  distribution of detectable FRBs depends on the assumed FRB models.

## 4. LUMINOSITY OF FRBS IN THE CASE THAT THEY ORIGINATE AT COSMOLOGICAL DISTANCES

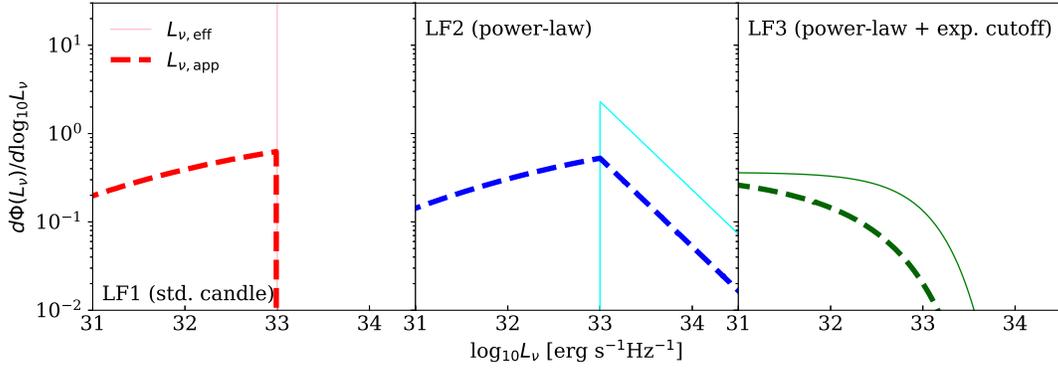
Here we determine the characteristic luminosity density of FRBs ( $L_{\nu,0}$ , see §3.3) that reproduces the observed  $\text{DM}_{\text{EX}}$  distribution best for each set of LF and  $\rho_{\text{FRB}}$  models assuming that FRBs originate at cosmological distances and the observed  $\text{DM}_{\text{EX}}$  is dominated by  $\text{DM}_{\text{IGM}}$ . We evaluate the goodness of fit using the Kolmogorov-Smirnov (KS) test. Figure 6 shows the KS test probability ( $P_{\text{KS}}$ ) that the observed sample can arise from the model distribution as a function of  $L_{\nu,0}$ , and figure 7 shows the best fit  $\text{DM}_{\text{IGM}}$  distributions.

Although a wide variety of LF and  $\rho_{\text{FRB}}$  models agree with the observed  $\text{DM}_{\text{EX}}$  distribution, the model with  $\rho_{\text{FRB}} \propto \text{SFR}$  plus LF2 does not reproduce the observations well. The best fit  $L_{\nu,0}$  for the  $\rho_{\text{FRB}} \propto \text{SFR}$  plus LF2 model is  $\log_{10}L_{\nu,0} [\text{erg s}^{-1}\text{Hz}^{-1}] \leq 31$  ( $P_{\text{KS}}$  is constant for smaller  $L_{\nu,0}$ ), which is smaller than the best fit values for the other models. This is because LF2 makes the  $\text{DM}_{\text{IGM}}$  distribution heavily tailed in the high DM end, while the observed  $\text{DM}_{\text{EX}}$  distribution steeply declines above  $\text{DM}_{\text{EX}} \gtrsim 1000 \text{ cm}^{-3}\text{pc}$  (the left panel of figure 5). The small  $L_{\nu,0}$  suppresses the high DM tail in the model distribution, and minimize the discrepancy between the model and observed distribution. However, it also overpredicts the number of FRBs with  $\text{DM}_{\text{EX}} \lesssim 500 \text{ cm}^{-3}\text{pc}$  making the model distribution broader than observed.

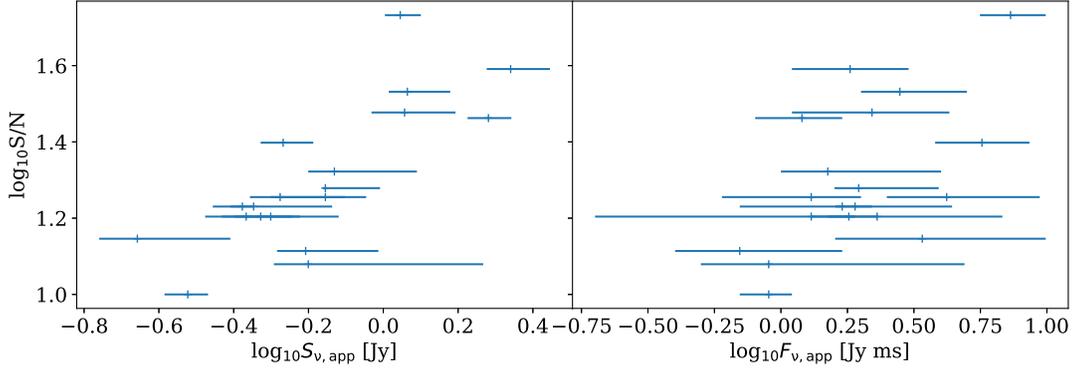
The discrepancy between the model with  $\rho_{\text{FRB}} \propto \text{SFR}$  plus LF2 and the observations suggests that neither an FRB LF with an extended bright-end without cutoff, nor an FRB LF that is dominated by its faint-end is favorable to reproduce the observed narrow  $\text{DM}_{\text{EX}}$  distribution when  $\rho_{\text{FRB}} \propto \text{SFR}$ , although the current FRB sample is not sufficient to rule out the model with high statistical significance. On the other hand, LF1 and LF3 reproduce the observations with similar  $L_{\nu,0}$  to each other ( $\log_{10}L_{\nu,0} [\text{erg s}^{-1}\text{Hz}^{-1}] \sim 34\text{--}35$ ), indicating that the faint-end of a  $L_{\nu,\text{eff}}$  PDF does not significantly affect the  $\text{DM}_{\text{EX}}$  distribution unless the  $L_{\nu,\text{eff}}$  PDF is dominated by its faint-end as in the case of LF2 with  $\log_{10}L_{\nu,0} [\text{erg s}^{-1}\text{Hz}^{-1}] \lesssim 31$ .

It is also noticeable that the  $\rho_{\text{FRB}} \propto \text{SFR}$  plus LF1 model produces sharp upper limit in the  $\text{DM}_{\text{IGM}}$  distribution which reflects the upper limit of the  $L_{\nu,\text{eff}}$  distribution making the agreement between the model and the observations poorer than those with the other models, although it is not rejected with a certain statistical significance. The decrease in the number of FRBs above  $\text{DM}_{\text{IGM}} \gtrsim 1000$  in the constant and  $M_{\star}$  models (the bottom panel of figure 1) can ease the conflict between LF1/LF2 and the observations. In those cases, LF2 also favors  $\log_{10}L_{\nu,0} [\text{erg s}^{-1}\text{Hz}^{-1}] \sim 34$ .

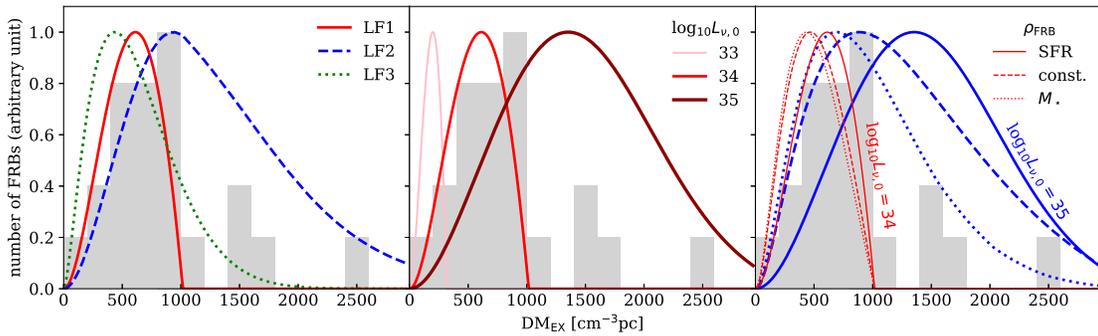
<sup>1</sup> <http://frbcat.org>



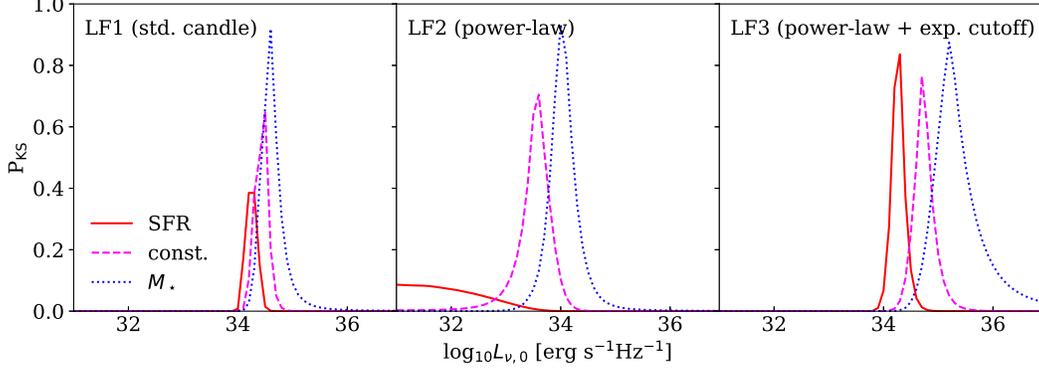
**Figure 3.** FRB luminosity function models considered in this study. The left, middle, and right panels show the PDFs of  $L_{\nu,\text{eff}}$  (thin solid) and  $L_{\nu,\text{app}}$  (thick dashed) for LF1 (standard candles), LF2 (power-law), and LF3 (power-law + exponential cutoff), respectively. The PDFs with  $\log_{10}L_{\nu,0}$  [ $\text{erg s}^{-1}\text{Hz}^{-1}$ ] = 33 are shown for each model.



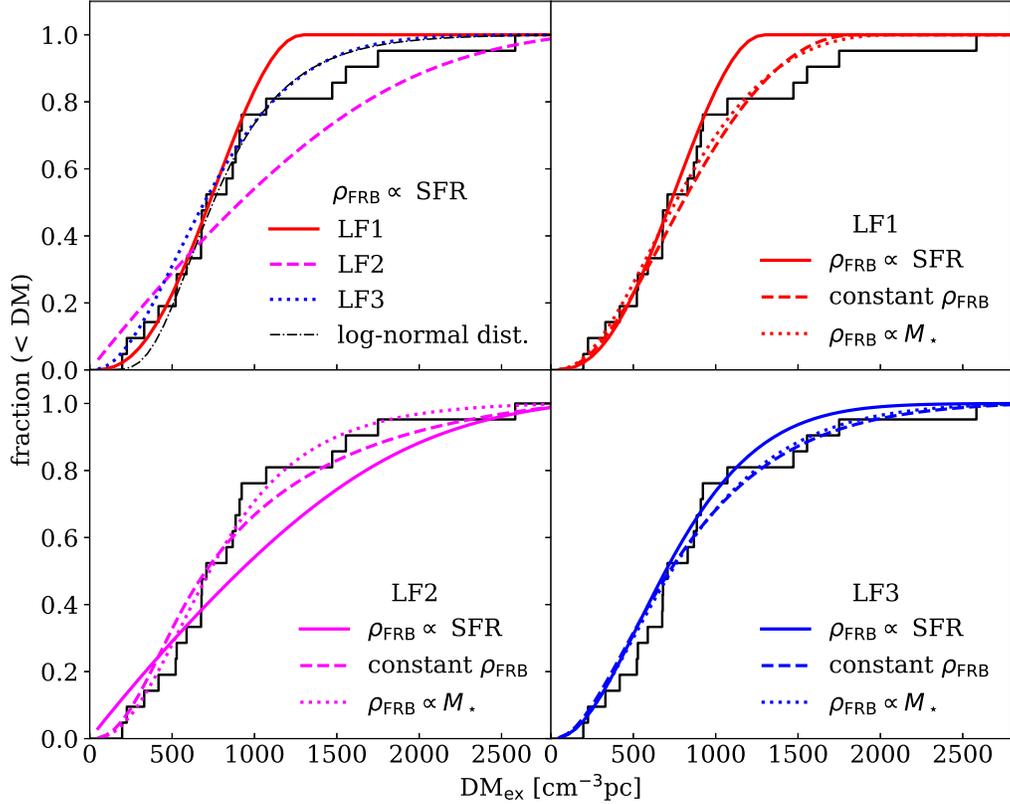
**Figure 4.** Left panel: the correlation between  $S_{\nu,\text{app}}$  and S/N in the Parkes sample. Right panel: same as the left panel but between  $F_{\nu,\text{app}}$  and S/N. The two peculiarly bright events, FRB 010724 and 150807, are excluded.



**Figure 5.** The  $DM_{\text{IGM}}$  distributions of detectable FRBs predicted with our models. LF1 with  $\log_{10}L_{\nu,0}$  [ $\text{erg s}^{-1}\text{Hz}^{-1}$ ] = 34, and the SFR model of  $\rho_{\text{FRB}}$  are used, unless otherwise stated. The DM distribution of the Parkes sample is shown together (gray histogram). Left panel: the predicted  $DM_{\text{IGM}}$  distributions with the three different LF models. Middle panel: the  $DM_{\text{IGM}}$  distributions with different  $L_{\nu,0}$ . Right panel: the  $DM_{\text{IGM}}$  distributions with the three different  $\rho_{\text{FRB}}$  models. Thin and thick lines indicate the distributions with  $\log_{10}L_{\nu,0}$  [ $\text{erg s}^{-1}\text{Hz}^{-1}$ ] = 34 and 35, respectively.



**Figure 6.**  $P_{KS}$  likelihood between the model and observed  $DM_{EX}$  distributions as a function of  $L_{\nu,0}$ . The left, middle, and right panels show  $P_{KS}$  for LF1, LF2, and LF3, respectively. The cases with the three  $\rho_{FRB}$  models are shown for each LF.



**Figure 7.** The cumulative  $DM_{EX}$  distribution of the Parkes FRB sample (histogram), and the best fit model distributions to the observations. Top left panel: the best fit models with LF1, LF2, and LF3 [solid (red), dashed (magenta), and dotted (blue) lines, respectively], the SFR model of  $\rho_{FRB}$  is assumed. A simple log-normal distribution with the median  $DM_{EX} = 750 \text{ cm}^{-3}\text{pc}$  and  $\sigma = 0.2 \text{ dex}$  is plotted together [dot-dashed line (black)]. Top right, bottom left, and bottom right panels: the best fit distributions with the different  $\rho_{FRB}$  models (SFR: solid line, constant: dashed line, and  $M_*$ : dotted line) with LF1, LF2, and LF3, respectively.

## 5. ARE FRBS COSMOLOGICAL OR LOCAL EVENTS?

Although the cosmological  $\text{DM}_{\text{IGM}}$  models (except that with  $\rho_{\text{FRB}} \propto \text{SFR}$  plus LF2) reproduce the observed  $\text{DM}_{\text{EX}}$  distribution well as previously shown by Dolag et al. (2015), Katz (2016), and Caleb et al. (2016), it should be noted that the distribution can also be explained by a simple log-normal distribution with the median  $\text{DM}_{\text{EX}} = 750 \text{ cm}^{-3}\text{pc}$  and  $\sigma = 0.2$  dex (shown in the top left panel of figure 7). Since a log-normal distribution is one of the most commonly seen PDFs in nature, it can be easily produced by a population of ionized gas in the circum/inter-stellar medium (CSM/ISM) around FRB sources without significant contribution from the IGM. Although  $\text{DM}_{\text{EX}}$  as high as  $750 \text{ cm}^{-3}\text{pc}$  is not likely to arise from ordinary galaxy ISM, if FRB sources are associated with ionized gas such as supernova remnant, it may significantly contribute to the observed  $\text{DM}_{\text{EX}}$  (Connor et al. 2016; Piro 2016; Murase et al. 2016; Lyutikov et al. 2016).

Here, we discuss how to distinguish whether FRBs are at cosmological distances (cosmological FRB model,  $\text{DM}_{\text{EX}}$  is dominated by  $\text{DM}_{\text{IGM}}$ ) or in the local universe (local FRB model,  $\text{DM}_{\text{EX}}$  is dominated by CSM/ISM in the vicinity of FRB sources).

### 5.1. $\log N$ - $\log S$ distribution

When a population of light source is homogeneously distributed in a Euclidean space as in the case of the local FRB model, observed flux density and fluence of the sources (so called  $\log N$ - $\log S$  distribution) follow the power-law distribution  $N(< S_\nu) \propto S_\nu^\alpha$  with index  $\alpha = -1.5$ . Although actual  $S_\nu$  and  $F_\nu$  of an FRB is not measurable due to the uncertain beam efficiency for each event,  $S_{\nu,\text{app}}$  and  $F_{\nu,\text{app}}$  would also follow the same power-law distribution when actual  $S_\nu$  and  $F_\nu$  follows the power-law distribution. Thus the observed distributions of  $S_{\nu,\text{app}}$  and  $F_{\nu,\text{app}}$  can serve as clues to distinguish whether FRBs are cosmological or local.

The earlier studies by Vedantham et al. (2016), Caleb et al. (2016), and Li et al. (2017) showed that the  $F_{\nu,\text{app}}$  distribution is flatter than the Euclidean case ( $\alpha > -1.5$ ). However, Macquart & Ekers (2018) pointed out that the  $F_{\nu,\text{app}}$  distribution of FRBs are largely affected by the detection incompleteness in the faint-end, and the steepness of the distribution is dependent on the fluence limit applied in the analysis. The recent analyses by Macquart & Ekers (2018) and Bhandari et al. (2017) showed that the observed FRB sample indicates that the  $F_{\nu,\text{app}}$  distribution is steeper than the Euclidean case ( $\alpha < -1.5$ ) above the fluence completeness limit of 2 Jy ms (Keane & Petroff

2015), although the current FRB sample size is not sufficient to reject the Euclidean case. On the other hand, Oppermann et al. (2016) examined the distribution of S/N of FRBs rather than  $S_\nu$  and  $F_\nu$ , and found that the  $\log N$ - $\log S$  distribution agrees well with the Euclidean case.

In the left panel of figure 8, we show the predicted  $S_{\nu,\text{app}}$  distributions by the cosmological FRB models assuming the SFR and  $M_\star$  models of  $\rho_{\text{FRB}}$  together with the three LF models. Hereafter, the parameter  $L_{\nu,0}$  in the LF models is fixed to the best fit value determined in §4, unless otherwise stated. The distribution functions predicted by the  $M_\star$  model of  $\rho_{\text{FRB}}$  are shallower than the Euclidean case regardless of the assumed LF model, while the distributions predicted by the SFR model of  $\rho_{\text{FRB}}$  are similar to the Euclidean case. This is because the cosmological expansion makes the  $\log N$ - $\log S$  distribution shallower, while larger source density at larger distance (as in the case of the SFR model of  $\rho_{\text{FRB}}$ ) makes the distribution steeper.

The right panel of figure 8 shows the same distribution as that in the left panel but for  $F_{\nu,\text{app}}$ . We have assumed the PDF of FRB energy follow the same formulations as the LF (LF1, LF2, and LF3), with the characteristic energy  $E_0 = L_{\nu,0} \times 3$  ms, and the fluence threshold to be 2 Jy ms which is the completeness limit derived by Keane & Petroff (2015) although many FRBs are detected below this fluence.

The predicted  $F_{\nu,\text{app}}$  distributions are steeper than the  $S_{\nu,\text{app}}$  distributions because fluence is not affected by the cosmological expansion of time. As a result, the  $F_{\nu,\text{app}}$  distribution functions predicted by the SFR model of  $\rho_{\text{FRB}}$  is steeper than the Euclidean case ( $\alpha \sim -1.8$ ), being consistent with the suggestions by the recent observations (Macquart & Ekers 2018; Bhandari et al. 2017). Although the Euclidean case ( $\alpha = -1.5$ ) is not fully ruled out by the current sample, if the steep fluence distribution is confirmed with the larger FRB sample, it indicates that FRBs originate at cosmological distances and  $\rho_{\text{FRB}}$  is larger at higher redshift resembling CSFH (see §6.2 for another possibility).

The difference of  $\alpha$  between the  $F_{\nu,\text{app}}$  distribution and the  $S_{\nu,\text{app}}$  distribution predicted by the cosmological FRB models with  $\rho_{\text{FRB}} \propto \text{SFR}$  can also reconcile the different  $\alpha$  suggested by Oppermann et al. (2016, S/N distribution) and Macquart & Ekers (2018,  $F_{\nu,\text{app}}$  distribution), given that  $S_{\nu,\text{app}}$  correlates well with S/N. On the other hand, the shallow  $\log N$ - $\log S$  distributions predicted by the  $M_\star$  model of  $\rho_{\text{FRB}}$  are close to the upper limit of  $\alpha$  derived by Amiri et al. (2017), and hence can be rejected in the near future.

In the current Parkes sample, 9 out of the 21 FRBs have larger  $F_{\nu, \text{app}}$  than the 2 Jy ms completeness limit. Macquart & Ekers (2018) examined how precisely  $\alpha$  can be determined for a variation of FRB sample sizes. Their results suggest that  $\sim 50$  FRBs above the fluence completeness limit would be necessary to distinguish  $\alpha = -1.8$  (our model prediction) from the Euclidean case with a statistical significance of  $\sim 95\%$ . If the fraction of FRBs above the fluence completeness limit in the observed sample remains unchanged, the total sample size required will be  $\sim 100$  FRBs.

### 5.2. Correlation between $\text{DM}_{\text{EX}}$ and $S_{\nu, \text{app}}$

Unlike  $F_{\nu, \text{app}}$ ,  $S_{\nu, \text{app}}$  correlates well with S/N and the cumulative distribution of  $S_{\nu, \text{app}}$  does not show significant incompleteness in its faint-end (figure 4 and 8). Hence we might be able to utilize larger observed sample when we investigate  $S_{\nu, \text{app}}$  rather than  $F_{\nu, \text{app}}$ . However, the cumulative distribution of  $S_{\nu, \text{app}}$  of the cosmological FRB model is similar to the Euclidean case (i.e., the local FRB model) when  $\rho_{\text{FRB}} \propto \text{SFR}$ , making it difficult to distinguish whether FRBs are cosmological or local solely with the  $S_{\nu, \text{app}}$  distribution.

One possible clue is the correlation between  $\text{DM}_{\text{EX}}$  and  $S_{\nu, \text{app}}$ . Yang et al. (2017) investigated the correlation between  $\text{DM}_{\text{EX}}$  and observed flux density to constrain the contribution of CSM/ISM in the vicinity of FRBs to  $\text{DM}_{\text{EX}}$  but without taking account of the  $\epsilon_{\text{beam}}$  variation within a receiver beam. Here we examine how efficiently the cosmological and local FRB models can be distinguished by the correlation between  $\text{DM}_{\text{EX}}$  and  $S_{\nu, \text{app}}$ , in the case  $\rho_{\text{FRB}} \propto \text{SFR}$ .

We compute distribution of FRBs on the parameter plane of  $\text{DM}_{\text{EX}}$  vs.  $S_{\nu, \text{app}}$  using the cosmological FRB models with LF1, LF2, and LF3 (the top three panels of figure 9). For the local FRB model, we assume that the  $\log N$ - $\log S$  distribution is the power-law with  $\alpha = -1.5$ , and  $\text{DM}_{\text{EX}}$  follows the log-normal distribution with the median  $\text{DM}_{\text{EX}} = 750 \text{ cm}^{-3} \text{ pc}$  and  $\sigma = 0.2 \text{ dex}$  (the bottom panel of figure 9).

We then randomly generate  $10^3$  sets of mock samples of  $\text{DM}_{\text{EX}}$  and  $S_{\nu, \text{app}}$  with sample size  $N_{\text{sample}}$  each in accordance with the model distributions, and compute probability distribution of the correlation coefficient between  $\text{DM}_{\text{EX}}$  and  $S_{\nu, \text{app}}$ . In figure 10, we show the mean and the standard deviation of the correlation coefficient distributions as functions of  $N_{\text{sample}}$ .

When the two peculiarly bright events, FRB 010724 and 150807, are excluded, the correlation coefficient between  $\text{DM}_{\text{EX}}$  and  $S_{\nu, \text{app}}$  in the Parkes sample is -0.35 with  $N_{\text{sample}} = 19$ . The correlation coefficient in the Parkes sample is already outside the standard deviation

of the local FRB model with the current  $N_{\text{sample}}$ . Although the correlation coefficient is still within the  $2\sigma$  error of the local FRB model, it can be ruled out if the same correlation coefficient is obtained with  $N_{\text{sample}} = 40$ .

Among the cosmological FRB models with  $\rho_{\text{FRB}} \propto \text{SFR}$ , LF3 agrees best with the observations. The correlation coefficient distribution with LF2 is hardly distinguishable from that with the local FRB model, however LF2 is disfavored by the  $\text{DM}_{\text{EX}}$  distribution (see §4). When the constant and  $M_*$  models of  $\rho_{\text{FRB}}$  are assumed, the correlation coefficient between  $\text{DM}_{\text{EX}}$  and  $S_{\nu, \text{app}}$  is not significantly changed with LF1 and LF3, while the model with LF2 shows the correlation coefficient of  $\sim -0.2$ – $-0.3$  depending on the  $\rho_{\text{FRB}}$  model.

If FRBs with higher  $\text{DM}_{\text{EX}}$  suffer more pulse broadening, it is possible that  $\text{DM}_{\text{EX}}$  and  $S_{\nu, \text{app}}$  correlates even in the local FRB model because pulse broadening may decrease  $S_{\nu, \text{app}}$ . However, we note that the pulse width of the FRBs in the Parkes sample is not correlated with their  $\text{DM}_{\text{EX}}$  (the correlation coefficient is -0.003).

## 6. DISCUSSION

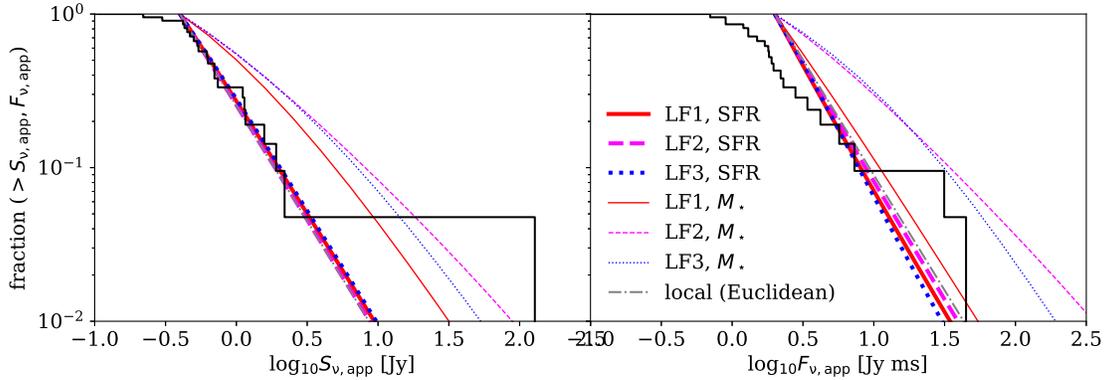
### 6.1. Inhomogeneous IGM

In the previous sections, we have assumed that the IGM density is spatially uniform at each redshift. McQuinn (2014) computed  $\text{DM}_{\text{IGM}}$  variation of FRBs at a single redshift caused by the inhomogeneity of the IGM. Their results show that the standard deviation of  $\text{DM}_{\text{IGM}}$  is 20–30% of the mean  $\text{DM}_{\text{IGM}}$  at each redshift in the range  $z \sim 0.3$ – $1.4$  when the spatial baryon distribution from a cosmological simulation is assumed.

To test how such  $\text{DM}_{\text{IGM}}$  variation affect the overall  $\text{DM}_{\text{IGM}}$  distribution that includes FRBs at various redshifts. We compute the  $\text{DM}_{\text{IGM}}$  distributions with the cosmological FRB models assuming that the probability distribution of  $\log_{10} \text{DM}_{\text{IGM}}$  at a redshift follows a Gaussian distribution with the mean value determined by equation (2) and  $\sigma = 0.1 \text{ dex}$ . We find that the inhomogeneity of the IGM does not significantly affect neither the overall  $\text{DM}_{\text{IGM}}$  distribution of FRBs, nor the PDF of the correlation coefficient between  $\text{DM}_{\text{EX}}$  and  $S_{\nu, \text{app}}$ . The  $\text{DM}_{\text{IGM}}$  distributions predicted with and without the inhomogeneity of the IGM are shown in figure 11.

### 6.2. $K$ -correction

We have also assumed that the  $K$ -correction factor to be  $\kappa_{\nu}(z) = 1$  (constant). The real  $\kappa_{\nu}(z)$  is determined by spectra of FRBs which is not well known yet (see equation 6). For example, when the spectrum of an FRB is a power-law,  $L_{\nu, \text{eff}}(\nu) \propto \nu^{\beta}$ , the  $K$ -correction



**Figure 8.** Left panel: the cumulative distribution of  $S_{\nu,\text{app}}$  ( $\log N$ - $\log S$ ), predicted by the cosmological FRB models with the SFR and  $M_*$  models of  $\rho_{\text{FRB}}$  (thick and thin lines, respectively). The results with the three LF models are shown (LF1, LF2, and LF3; solid, dashed, and dotted lines, respectively). The  $S_{\nu,\text{app}}$  distribution of the Parkes sample is plotted together (histogram), and the dot-dashed line indicates the distribution in the Euclidean case ( $\alpha = -1.5$ ). Right panel: same as the left panel but for  $F_{\nu,\text{app}}$ .

factor is  $\kappa_\nu(z) = (1+z)^\beta$ . If  $\kappa_\nu(z)$  increases with redshift, we would detect more FRBs at higher redshifts. In this sense, there is a degeneracy between the  $K$ -correction (spectrum) and  $\rho_{\text{FRB}}$ . In figure 12, we show the  $\text{DM}_{\text{IGM}}$  distributions with  $\beta = -2, 0$ , and  $2$ , assuming LF1,  $\rho_{\text{FRB}} \propto \text{SFR}$ , and  $\log_{10} L_{\nu,0}$  [ $\text{erg s}^{-1} \text{Hz}^{-1}$ ] =  $34$ .

We have determined the  $L_{\nu,0}$  that reproduces the observed  $\text{DM}_{\text{EX}}$  distribution with  $\beta = \pm 2$  following the same procedure as in §4 (figure 13). The observed  $\text{DM}_{\text{EX}}$  distribution can be reproduced in a wide variety of cases but with different  $L_{\nu,0}$ . Once the best fit  $L_{\nu,0}$  for each  $\beta$  is determined, the  $K$ -correction does not significantly affect the correlation coefficient between  $\text{DM}_{\text{EX}}$  and  $S_{\nu,\text{app}}$ . However, we note that  $\beta > 0$  can also make the cumulative distribution of  $F_{\nu,\text{app}}$  steeper as well as the increase of  $\rho_{\text{FRB}}$  at high redshifts, due to the degeneracy between the  $K$ -correction and  $\rho_{\text{FRB}}$ . Observations with different  $\nu_{\text{obs}}$  are important to break the degeneracy.

## 7. CONCLUSIONS

We have computed the  $\text{DM}_{\text{EX}}$  distribution, the  $\log N$ - $\log S$  distribution, and the  $\text{DM}_{\text{EX}}-S_{\nu,\text{app}}$  correlation based on the analytic models of cosmic rate density history ( $\rho_{\text{FRB}}$ ) and LF of FRBs. Comparing the model predictions with the observations, we have found that

the cumulative distribution of apparent fluxes suggests that FRBs are at cosmological distances with higher  $\rho_{\text{FRB}}$  at higher redshifts resembling CSFH (or FRBs typically have very hard radio spectra with  $L_\nu$  larger at higher frequency, i.e.,  $\beta > 0$ ), although the sample size of the current observations is not sufficient to rule out that FRBs originate in the local universe.

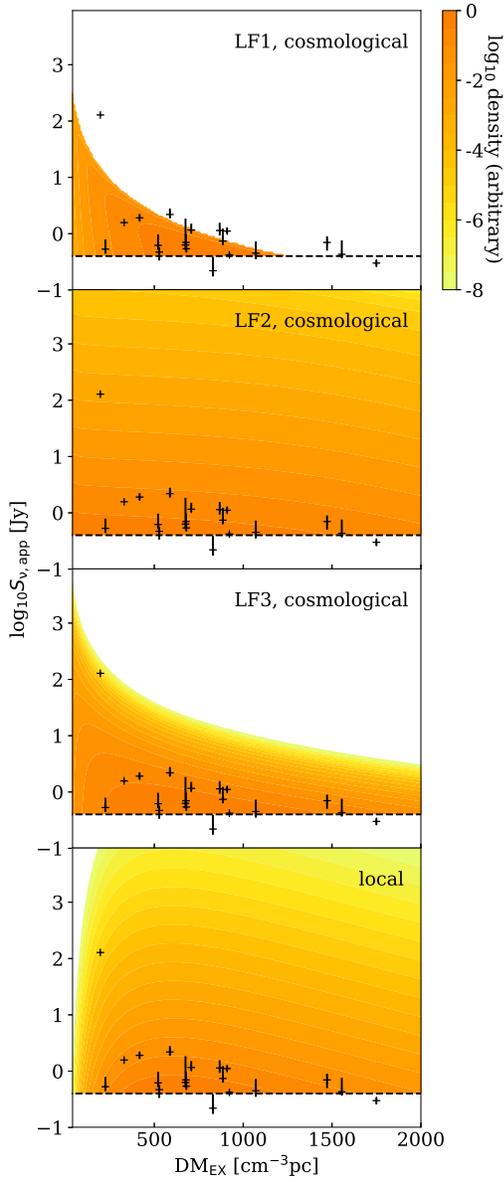
If  $\rho_{\text{FRB}}$  is proportional to SFR, the narrow  $\text{DM}_{\text{EX}}$  distribution of the observed FRBs favors an FRB LF with a bright-end cutoff at  $\log_{10} L_\nu$  [ $\text{erg s}^{-1} \text{Hz}^{-1}$ ]  $\sim 34$ . Although the constraint on the faint-end of FRB LF is rather weak, an FRB LF that is dominated by its faint-end is also disfavored. However, the statistical significance of the constraint with the current sample is still low.

The correlation coefficient between  $\text{DM}_{\text{EX}}$  and  $S_{\nu,\text{app}}$  is potentially a very powerful tool to distinguish whether FRBs are at cosmological distances or in the local universe more robustly with future observations, which may provide us with higher statistical significance of the constraint than the  $\log N$ - $\log S$  distribution for a given sample size.

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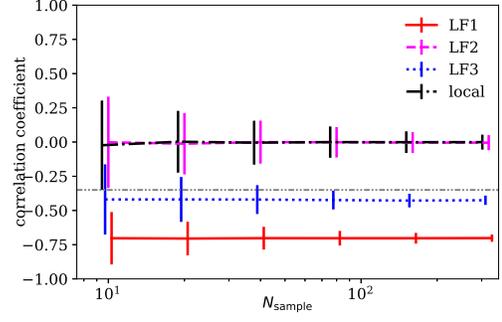


**Figure 9.** Distributions of FRBs on the parameter plane of  $DM_{\text{EX}}$  vs.  $S_{\nu, \text{app}}$ . The top three panels show the distributions of the cosmological FRBs with LF1, LF2, and LF3, respectively.  $\rho_{\text{FRB}}$  is assumed to be proportional to SFR. The bottom panel shows the distribution of the local FRB model. The horizontal dashed line indicates the assumed detection limit in our model (0.4 Jy). FRBs in the Parkes sample are overplotted with crosses.

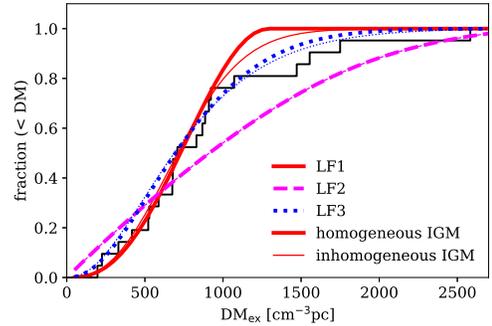
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**Figure 10.** Mean and standard deviation of the correlation coefficient between  $DM_{\text{EX}}$  and  $S_{\nu, \text{app}}$  generated by the Monte Carlo tests according to the distribution function shown in figure 9. Datapoints connected with solid, dashed, dotted lines show the correlation coefficient distribution with the cosmological FRB models with LF1, LF2, and LF3, respectively. The correlation coefficient distribution with the local FRB model is shown with datapoints connected with dot-dashed line. The datapoints are slightly shifted sideways for visibility. The horizontal double-dot-dashed line indicates the correlation coefficient between  $DM_{\text{EX}}$  and  $S_{\nu, \text{app}}$  in the Parkes sample with FRB 010724 and 150807 excluded ( $N_{\text{sample}} = 19$ ). The random generation of mock sample is performed 1000 times for each  $N_{\text{sample}}$ .

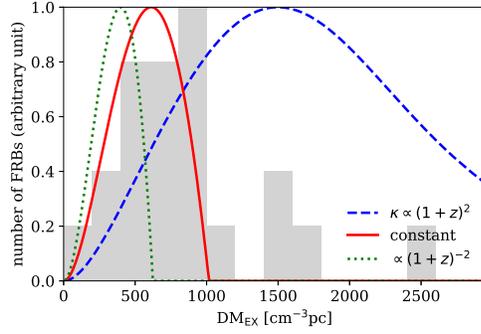


**Figure 11.** The cumulative  $DM_{\text{EX}}$  distributions predicted with (thin line) and without (thick line) the inhomogeneity of the IGM. The solid, dashed, and dotted lines represent the distributions predicted with LF1, LF2, and LF3, respectively. The SFR model of  $\rho_{\text{FRB}}$  is assumed. The distributions without the inhomogeneity are identical to those in figure 7.  $DM_{\text{EX}}$  of the Parkes sample is plotted together (histogram).

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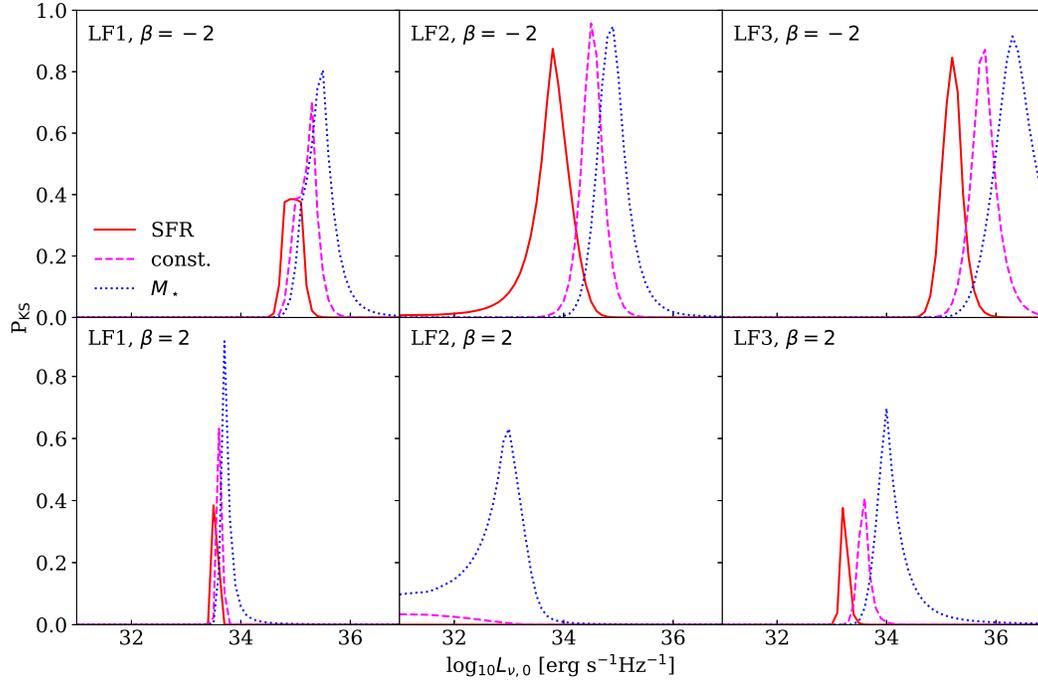
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**Figure 12.** Same as figure 5, but with  $\beta = -2, 0,$  and  $2$ . LF1 with  $\log_{10} L_{\nu,0} [\text{erg s}^{-1} \text{Hz}^{-1}] = 34$ , and the SFR model of  $\rho_{\text{FRB}}$  are assumed.

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**Figure 13.** Same as figure 6, but with  $\beta = \pm 2$ .